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Asymptotic Error Estimates for Quintic Spline-on-Spline Interpolation

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1. INTRODUCTION

We consider a quintic spline-on-spline technique for approximating the second (or higher order) derivative of a function from its values on a uniform partition of $I = [0, 1]$ with knots $x_i = ih$, $i = 0(1)n$. For a cubic spline, there is computational evidence that a cubic spline-on-spline interpolation gives excellent results for the second derivative of $\sin x$ ([1]). Dolezal and Tewarson [2] obtained error bounds for the interpolation, and we derived asymptotic expansions of the errors in the second and third derivatives ([6]). Let s be an interpolatory cubic spline of a sufficiently smooth function f and let p be a cubic spline-on-spline interpolant of the derivative of s. Then we have under appropriate end conditions,

$$f_i'' - p_i' = (h^4/90) f_i^{(6)} + \cdots \tag{1}$$

where $f_i^{(k)} = f^{(k)}(ih)$ and $s_i^{(k)} = s^{(k)}(ih)$ ([6]).

The object of this paper is to derive analogous asymptotic error estimates for the quintic spline-on-spline interpolation. Let us denote an interpolating quintic spline to f by s and a quintic spline-on-spline interpolant of s' by p:

$$\begin{aligned} &\text{(i)} \quad s_i = f_i, \qquad i = 0(1)n \\ &\text{(ii)} \quad p_i = s_i', \qquad i = 0(1)n. \end{aligned} \tag{2}$$

In Section 3, we shall prove the following asymptotic expansions of the errors under end conditions (20):

$$\begin{aligned} &\text{(i)} \quad f_i'' - s_i'' = -(h^4/720) f_i^{(6)} + \cdots \\ &\text{(ii)} \quad f_i'' - p_i' = -(h^6/2520) f_i^{(8)} + \cdots. \end{aligned} \tag{3}$$

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By means of the asymptotic expansion 3(i), Richardson type extrapolation is used to get an $O(h^6)$ second derivative estimate without using the quintic spline-on-spline technique:

$$f_i'' - (1/15)\{16 s_{h/2}''(x_i) - s_h''(x_i)\} = -(h^6/67200) f_i^{(8)} + \cdots \tag{4}$$

for any mesh point x_i bounded away from the endpoints where $s_h(x)$ and $s_{h/2}(x)$ are quintic spline interpolations to f with uniform mesh sizes h and $h/2$, respectively ([7]).

On the other hand, by 3(ii) we have

$$f_i'' - p_{h/2}'(x_i) = -(h^6/161280) f_i^{(8)} + \cdots \tag{5}$$

where $p_{h/2}(x)$ is a quintic spline-on-spline interpolation to f with uniform mesh size $h/2$.

Since $67200/161280 = 5/12$, spline-on-spline technique yields better estimate than extrapolation. As for computational effort, we may have the following result. In extrapolation method, we have to solve two linear systems of order n and $2n$ to determine s_h and $s_{h/2}$, respectively. In spline-on-spline technique, the coefficient matrices for determining $s_{h/2}$ and $p_{h/2}$ are exactly the same, and so $p_{h/2}$ is easily determined with little additional effort.

Hence we may have a justification for using the quintic spline-on-spline technique instead of extrapolation method.

In the last section some numerical results are given.

2. Some Lemmas

In the present paper we consider end conditions of the form:

$$s_0^{(4)} + \alpha_1 s_1^{(4)} + \beta_1 s_2^{(4)} = c_0, \qquad s_0^{(4)} + \gamma_1 s_1^{(4)} + \delta_1 s_2^{(4)} + \eta_1 s_3^{(4)} = c_1,$$

$$s_n^{(4)} + \alpha_2 s_{n-1}^{(4)} + \beta_2 s_{n-2}^{(4)} = c_n,$$

$$s_n^{(4)} + \gamma_2 s_{n-1}^{(4)} + \delta_2 s_{n-2}^{(4)} + \eta_2 s_{n-3}^{(4)} = c_{n-1}.$$

Let θ and κ ($|\theta| > |\kappa| > 1$) be the solutions of the quartix polynomial: $x^4 + 26x^3 + 66x^2 + 26x + 1 = 0$, and $p_i(x) = 1 + \alpha_i x + \beta_i x^2$ and $q_i(x) = 1 + \gamma_i x + \delta_i x^2 + \eta_i x^3$. In what follows, for any finite dimensional vector and matrix, let us denote their maximum norms by $|\cdot|$. Now we shall prove the following two lemmas.

Lemma 1. *If $p_i(1/\theta)\, q_i(1/\kappa) - p_i(1/\kappa)\, q_i(1/\theta) \neq 0$, $i = 1, 2$, the following n*

by n matrix A_n of band-width five is nonsingular for sufficiently large n and in addition

$$|A_n^{-1}| \leqslant C$$

where C is a generic constant independent of n and

$$A_n = \begin{bmatrix} 1 & \alpha_1 & \beta_1 & & & & \\ 1 & \gamma_1 & \delta_1 & \eta_1 & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 26 & 66 & 26 & 1 \\ & & & \eta_2 & \delta_2 & \gamma_2 & 1 \\ & & & & \beta_2 & \alpha_2 & 1 \end{bmatrix}. \tag{6}$$

Proof. Let us consider a linear system:

$$A_{n+1}\xi = \lambda \tag{7}$$

where $\xi = (\xi_0, \xi_1, ..., \xi_n)^T$ and $\lambda = (\lambda_0, \lambda_1, ..., \lambda_n)^T$.

$$\begin{cases} 66\xi_2 + 26\xi_3 + \xi_4 = \lambda_2 - 26\xi_1 - \xi_0 \\ 26\xi_2 + 66\xi_3 + 26\xi_4 + \xi_5 = \lambda_3 - \xi_1 \\ \xi_{i-2} + 26\xi_{i-1} + 66\xi_i + 26\xi_{i+1} + \xi_{i+2} = \lambda_i \\ \qquad i = 4(1)n-4 \\ 26\xi_{n-2} + 66\xi_{n-3} + 26\xi_{n-4} + \xi_{n-5} = \lambda_{n-3} - \xi_{n-1} \\ 66\xi_{n-2} + 26\xi_{n-3} + \xi_{n-4} = \lambda_{n-2} - 26\xi_{n-1} - \xi_n. \end{cases} \tag{8}$$

Let $D = (d_{i,j},\ 2 \leqslant i,\ j \leqslant n-2)$ be the inverse of the above diagonally dominant coefficient matrix. Then we have

$$\begin{aligned} \xi_i = \sum_{j=2}^{n-2} d_{i,j}\lambda_j - d_{i,2}(26\xi_1 + \xi_0) - d_{i,3}\xi_1 \\ - d_{i,n-3}\xi_{n-1} - d_{i,n-2}(26\xi_{n-1} + \xi_n) \\ i = 2(1)n-2. \end{aligned} \tag{9}$$

Here we shall require the following properties of the coefficient matrix D:

(i) $\sum_{j=2}^{n-2} |d_{i,j}| \leqslant 1/12, \qquad i = 2(1)n-2 \qquad$ ([1])

(ii) $d_{2,2}=1/(\theta\kappa)+O(|\kappa|^{-n}),$

$$d_{3,2}=1/(\theta\kappa)(1/\theta+1/\kappa)+O(|\kappa|^{-n}), \tag{10}$$

$$d_{3,3}=-\{1/\theta^2+1/(\theta\kappa)+1/\kappa^2+26/(\theta\kappa)(1/\theta+1/\kappa)\}+O(|\kappa|^{-n})$$

(iii) $d_{i,2}, d_{i,3}=O(|\kappa|^{-i}), \qquad i=2(1)n-2.$

The above properties are easily obtained by solving the difference equations, i.e.,

$$d_{i,2}=a\theta^{i-2}+b\theta^{2-i}+c\kappa^{i-2}+d\kappa^{2-i}, \qquad i=2(1)n-2 \tag{11}$$

where coefficients (a, b, c, d) are the solutions of linear equations:

$$\begin{cases} p(\theta)a+p(1/\theta)b+p(\kappa)c+p(1/\kappa)d=1 \\ q(\theta)a+q(1/\theta)b+q(\kappa)c+q(1/\kappa)d=0 \\ \theta^{n-4}q(1/\theta)a+\theta^{4-n}q(\theta)b+\kappa^{n-4}q(1/\kappa)c+\kappa^{4-n}q(\kappa)d=0 \\ \theta^{n-4}p(1/\theta)a+\theta^{4-n}p(\theta)b+\kappa^{n-4}p(1/\kappa)c+\kappa^{4-n}p(\kappa)d=0 \end{cases} \tag{12}$$

$(p(x)=x^2+26x+66$ and $q(x)=x^3+26x^2+66x+26)$. Since "the determinant of the above coefficient matrix" becomes $(\theta\kappa)^{n-4}\{p(1/\theta)\,q(1/\kappa)-p(1/\kappa)\,q(1/\theta)\}^2+\cdots=(\theta\kappa)^{n-2}(\theta-\kappa)^2+\cdots$, we have

$$a=O(|\theta|^{-2n}), \qquad c=O(|\theta\kappa|^{-n}),$$

$$b=-1/\{\theta(\theta-\kappa)\}+O(|\kappa|^{-n}) \qquad \text{and} \qquad d=1/\{\kappa(\theta-\kappa)\}+O(|\kappa|^{-n}).$$

Using the above asymptotic estimates, we have

$$d_{i,2}=(\kappa^{1-i}-\theta^{1-i})/(\theta-\kappa)+O(|\kappa|^{-n}), \qquad i=2(1)n-2. \tag{13}$$

Similarly we have

$$d_{i,3}=\{(\kappa+26)\,\theta^{1-i}-(\theta+26)\,\kappa^{1-i}\}/(\theta-\kappa)$$
$$+\ O(|\kappa|^{-n}), \qquad i=2(1)n-2. \tag{14}$$

This completes the derivation of properties (10ii) and (10iii).

Now we return to the proof of Lemma 1. Substituting ξ_i, $i=2$, 3, $n-3$ and $n-2$ represented by equations (9) into the first and last two equations of (6) yields

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \xi_0 \\ \xi_1 \\ \xi_{n-1} \\ \xi_n \end{bmatrix} = \begin{bmatrix} \bar{\lambda}_0 \\ \bar{\lambda}_1 \\ \bar{\lambda}_{n-1} \\ \bar{\lambda}_n \end{bmatrix}. \tag{15}$$

Here, in virtue of (10), we have

(i) $\bar{\lambda}_i$, $i=0$, 1, $n-1$ and n are some linear combinations of λ_j, $j=0(1)n$ such that

$$|\bar{\lambda}_i| \leqslant C\,|\lambda|.$$

(ii)

$$A_{1,2}, A_{2,1} = O(|\kappa|^{-n}) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

(iii)

$$A_{i,i} = \begin{bmatrix} 1 - d_{2,2}\beta_i & \alpha_i - (26d_{2,2} + d_{2,3})\,\beta_i \\ 1 - d_{2,2}\delta_i - d_{3,2}\eta_i & \gamma_i - (26d_{2,2} + d_{2,3})\,\delta_i \\ & -(26d_{3,2} + d_{3,3})\,\eta_i \end{bmatrix}.$$

$$i = 1, 2.$$

By (10ii) and (10iii), we have

$$\det(A_{i,i}) = (1/\kappa - 1/\theta)^{-1}\{p_i(1/\theta)\,q_i(1/\kappa) - p_i(1/\kappa)\,q_i(1/\theta)\} + \cdots \neq 0, \qquad i = 1, 2 \tag{16}$$

for sufficiently large n. Hence, by (15) we have an inequality of the form

$$|\xi_i| \leqslant C\,|\lambda|, \qquad i = 0, 1, n-1 \text{ and } n. \tag{17}$$

From (9), by (17) and (10i) we have

$$|\xi| \leqslant C\,|\lambda| \qquad \text{for sufficiently large } n.$$

By (7), this inequality implies the nonsingularity of A_{n+1} for sufficiently large n and in addition

$$|A_{n+1}^{-1}| \leqslant C. \tag{18}$$

This completes the proof of this Lemma 1.

Similarly as in the proof of Lemma 1 we have the following lemma that is required for the error estimates at any mesh point bounded away from the endpoints.

LEMMA 2 (cf. [4]). *Let us denote the* (i, j)*-component of the inverse of* A_n *in Lemma* 1 *by* $(A_n^{-1})_{i,j}$. *Then we have*

$$(A_n^{-1})_{i,0}, (A_n^{-1})_{i,1} = O(|\kappa|^{-i} + |\kappa|^{i-n}), \qquad i = 0(1)n \tag{19}$$

for sufficiently large n.

3. ASYMPTOTIC ERROR ESTIMATES

Since quintic splines s and p depend upon $n+5$ parameters, four additional end conditions are required toward the determination of these, respectively. In the present paper, we take these to be homogeneous end conditions:

$$\begin{aligned} &\text{(i)} \quad \Delta^r s_0^{(4)} = \Delta^{r+1} s_0^{(4)} = \nabla^r s_n^{(4)} = \nabla^{r+1} s_n^{(4)} = 0 \\ &\text{(ii)} \quad \Delta^r p_0^{(4)} = \Delta^{r+1} p_0^{(4)} = \nabla^r p_n^{(4)} = \nabla^{r+1} p_n^{(4)} = 0 \end{aligned} \tag{20}$$

where $r = 5$ or 6 or 7, Δ and ∇ are forward and backward difference operators, respectively. By repeated use of the consistency relation for quintic spline:

$$\begin{aligned} &(1/120)(s_{i+2}^{(4)} + 26 s_{i+1}^{(4)} + 66 s_i^{(4)} + 26 s_{i-1}^{(4)} + s_{i-2}^{(4)}) \\ &\quad = (1/h^4)(s_{i+2} - 4s_{i+1} + 6s_i - 4s_{i-1} + s_{i-2}), \end{aligned} \tag{21}$$

condition $\Delta^r s_0^{(4)} = 0$ $(r \neq 4)$ may be rewritten as follows

$$s_0^{(4)} + a_r s_1^{(4)} + b_r s_2^{(4)} + c_r s_3^{(4)} = L_r(s_0, s_1, ..., s_r) \tag{22}$$

where a_r, b_r and c_r are real constants and L_r is some linear combination of s_j, $j = 0(1)r$ (Table I). For example,

$$L_6 = (1/317)(19021 g_2 - 813 g_3 + 33 g_4 - g_5)$$

$$L_7 = (1/3840)(460801 g_2 - 19834 g_3 + 846 g_4 - 34 g_5 + g_6)$$

TABLE I

r	5	6	7	8
a_r	27	26	8229/317	59805/2304
b_r	67	65	20571/317	149490/2304
c_r	25	304/13	7363/317	53469/2304

where we denote the right-hand side of (21) by g_i. By (2i), (21) and (22), we have a system of $s_i^{(4)}$, $i=0(1)n$, whose coefficient matrix A_{n+1} is almost of band-width five:

$$A_{n+1}=\begin{bmatrix} 1 & a_{r+1} & b_{r+1} & c_{r+1} & & & \\ 1 & a_r & b_r & c_r & & & \\ 1 & 26 & 66 & 26 & 1 & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & 1 & 26 & 66 & 26 & 1 \\ & & & c_r & b_r & a_r & 1 \\ & & & c_{r+1} & b_{r+1} & a_{r+1} & 1 \end{bmatrix}. \tag{23}$$

By Taylor series expansion, we have

$$\begin{aligned}(1/120)\,A_{n+1}(e_0^{(4)}, e_1^{(4)},\ldots, e_n^{(4)})^T &= (O(h^{r+1}), O(h^r), (h^2/12)f_2^{(6)}+(h^4/60)f_i^{(8)} \\ &\quad + \cdots, O(h^r), O(h^{r+1}))^T\end{aligned} \tag{24}$$

where

$$e_i^{(4)}=f_i^{(4)}-s_i^{(4)}, \qquad i=0(1)n.$$

After eliminating $(1, 4)$ and $(n+1, n-2)$-components of (24), by Lemma 1 we have

$$\begin{aligned}f_i^{(4)}-s_i^{(4)} &= (h^2/12)f_i^{(6)}-(h^4/240)f_i^{(8)} \\ &\quad + O(h^{\min(6,r)}), \qquad i=0(1)n.\end{aligned} \tag{25}$$

Since

$$\begin{aligned}&\text{(i)} \quad s_i''=(1/h^2)(2s_i-5s_{i+1}+4s_{i+2}-s_{i+3}) \\ &\qquad + (h^2/120)(18s_i^{(4)}+65s_{i+1}^{(4)}+26s_{i+2}^{(4)}+s_{i+3}^{(4)}), \\ &\text{(ii)} \quad s_i'=1/(6h)(-11s_i+18s_{i+1}-9s_{i+2}+2s_{i+3}) \\ &\qquad - (h^3/720)(19s_i^{(4)}+108s_{i+1}^{(4)}+51s_{i+2}^{(4)}+2s_{i+3}^{(4)}) \qquad ([3]),\end{aligned} \tag{26}$$

we have

$$\begin{aligned}&\text{(i)} \quad f_i''-s_i'' = -(h^4/720)f_i^{(6)}+(h^6/3360)f_i^{(8)} \\ &\qquad + O(h^{\min(8,r+2)}), \qquad i=0(1)n \\ &\text{(ii)} \quad f_i'-s_i' = -(h^6/5040)f_i^{(7)}+O(h^{\min(8,r+3)}), \qquad i=0(1)n.\end{aligned} \tag{27}$$

This completes the proof of (3i).

Next we shall derive (3ii). Since p is also quintic, in virtue of the consistency relation and (2ii), we have

$$\begin{aligned}(1/120)(p^{(4)}_{i+2}+26p^{(4)}_{i+1}+66p^{(4)}_{i}+26p^{(4)}_{i-1}+p^{(4)}_{i-2})\\ =(1/h^4)(p_{i+2}-4p_{i+1}+6p_i-4p_{i-1}+p_{i-2})\\ =(1/h^4)(s'_{i+2}-4s'_{i+1}+6s'_i-4s'_{i-1}+s'_{i-2}).\end{aligned} \tag{28}$$

By means of the consistency relation for quintic spline s:

$$\begin{aligned}(1/h^4)(s'_{i+2}-4s'_{i+1}+6s'_i-4s'_{i-1}+s'_{i-2})\\ =1/(24h)(s^{(4)}_{i+2}+10s^{(4)}_{i+1}-10s^{(4)}_{i-1}-s^{(4)}_{i-2}),\end{aligned}$$

the right-hand side of (28) may be easily determined by using the already obtained $s_i^{(4)}$, $i=0(1)n$. By (20ii) and (28), we have a system of equations of $p_i^{(4)}$, $i=0(1)n$, whose coefficient matrix is exactly the same A_{n+1} for determining $s_i^{(4)}$, $i=0(1)n$. That is, $p_i^{(4)}$, $i=0(1)n$ are very easily determined with little additional effort. Similarly as for s, using again Lemma 1 yields

$$f_i^{(5)}-p_i^{(4)}=(h^2/12)f_i^{(7)}+O(h^{\min(4,r-1)}),\qquad i=0(1)n. \tag{29}$$

Since $p_i=s'_i$, by the consistency relation (26ii) we have

$$\begin{aligned}p'_i=(1/6h)(-11s'_i+18s'_{i+1}-9s'_{i+2}+2s'_{i+3})\\ -(h^3/720)(19p_i^{(4)}+108p^{(4)}_{i+1}+51p^{(4)}_{i+2}+2p^{(4)}_{i+3}).\end{aligned} \tag{30}$$

By (27ii), (29) and (30), we have

$$f''_i-p'_i=-(h^6/2520)f_i^{(8)}+O(h^{\min(8,r+2)}),\qquad i=0(1)n. \tag{31}$$

Thus we have

THEOREM 1. *Let s and p be quintic interpolants of f and s' on a uniform partition of I, respectively. Then we have under end conditions* (20):

$$\begin{aligned}&\text{(i)}\quad f''_i-s''_i=-(h^4/720)f_i^{(6)}+(h^6/3360)f_i^{(8)}\\ &\qquad\qquad +O(h^{\min(8,r+2)}),\qquad i=0(1)n,\\ &\text{(ii)}\quad f''_i-p'_i=-(h^6/2520)f_i^{(8)}+O(h^{\min(8,r+2)}),\qquad i=0(1)n.\end{aligned} \tag{32}$$

Using Lemma 2 (i.e., Kershaw's technique in [4]), we have

TABLE II

($f(x)=e^{5x}$)

x	$e_1(x)$	$e_2(x)$	$e_3(x)$
0	−0.215(−4)[a]	−0.951(−6)	−0.650(−4)
$\frac{1}{2}$	−0.251(−3)	−0.175(−5)	−0.422(−5)
1	−0.302(−2)	0.123(−4)	−0.709(−3)

[a] We denote -0.215×10^{-4} by $-0.215(-4)$.

THEOREM 2. *For any integer* $4\leqslant r\leqslant 6$, *we have*

$$\text{(i)}\quad f_i''-s_i''=-(h^4/720)f_i^{(6)}+O(h^6),\qquad i=0(1)n,$$

$$\text{(ii)}\quad f_i''-p_i'=-(h^6/2520)f_i^{(8)}+O(h^8),\qquad i=0(1)n$$

for any mesh point bounded away from the endpoints $x=0$ *and* $x=1$.

4. NUMERICAL ILLUSTRATION

The results of some numerical computational experiments are given in Tables II and III for the functions e^{5x} and $\log(1+x)$. We choose $(h,r)=(1/16,7)$ and denote

$$e_1(x)=f''(x)-s''(x),\qquad e_2(x)=f''(x)-p_{h/2}'(x)$$

$$e_3(x)=f''(x)-(1/15)\{16s_{h/2}''(x)-s_h''(x)\}.$$

From above, we have

$$e_2(\tfrac{1}{2})/e_3(\tfrac{1}{2})\doteqdot 0.415\qquad (e^{5x})$$

$$\doteqdot 0.413\qquad (\log(1+x))$$

which correspond with the predicted value $5/12\doteqdot 0.417$.

TABLE III

($f(x)=\log(1+x)$)

x	$e_1(x)$	$e_2(x)$	$e_3(x)$
0	0.148(−6)	−0.688(−8)	0.102(−6)
$\frac{1}{2}$	0.139(−7)	0.726(−10)	0.176(−9)
1	0.252(−8)	0.423(−10)	−0.317(−8)

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